Pergamon

Int. J. Solids Structures Vol. 34, No. 11, pp. 1293–1305, 1997

0020-7683/97 $17.00 + .00

PII: S0020-7683(96)00134-5

DOUBLY ASYMPTOTIC APPROXIMATIONS FOR TRANSIENT ELASTODYNAMICS

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(*Received* 8 *February* 1996; *in revised form* 8 *July* 1996)

Abstract—A doubly asymptotic approximation (DAA) is an approximate temporal impedance relation at the boundary of a continuous medium; it approaches exactness at both early and late times, effecting a smooth transition between. Here, first- and second-order DAAs are derived for a uniform, isotropic, elastic medium of either infinite or semi-infinite extent. The derivations proceed from pertinent singly asymptotic approximations and employ the method of operator matching previously used for acoustic domains. A simple problem with spherical symmetry is considered that illustrates the characteristics of the singly and doubly asymptotic approximations.

INTRODUCTION

Materials characterization, flaw detection, medical diagnosis, earthquake-resistant construction, oil exploration and defense technology are some of the areas in which dynamic boundary-element analysis has been productively used. Considerable progress has been made in time-harmonic applications, because the pertinent integral operators admit satisfactorily accurate and efficient computation. However, computational impracticality has hindered the application of transient boundary-element analysis, which has motivated the development of approximate approaches.

The focus here is on a body embedded in an unbounded, uniform, isotropic, linear-elastic medium, either infinite or semi-infinite. The body itself, and/or some medium in the immediate vicinity of the body, may exhibit nonlinear behavior; in the latter case, the boundary on which the integral operators act encloses both the body and the nonlinearly behaving medium.

In a radiation problem, the total elastodynamic field outside the integral-operator boundary is the radiated field. In a scattering problem, the total field is separated into the (known) incident field and the (unknown) scattered field. The incident field is that which would exist if the region inside the boundary were replaced by linear-elastic medium. The scattered field, then, is merely the difference, at any point and any time, between the total field and the incident field, i.e., it is the field caused by the presence of the scatterer.

Analysis of a transient radiated or scattered field on an integral-operator boundary is greatly facilitated by the formulation of a *temporal impedance relation* (TIR) that provides the body's time-domain view of the surrounding medium (Geers and Zhang, 1994). When the TIR is combined with the equations of motion for the body, the interface compatibility conditions at the boundary, and the pertinent initial and forcing conditions, a complete mathematical formulation is obtained that lends itself to numerical solution.

As indicated above, exact TIRs are complicated and costly to employ in computations, as they are nonlocal in both space and time, i.e., they require full computational matrices and long-memory response data. Hence, accurate approximate TIR's are needed for efficient computation. This paper is about approximate TIRs that approach exactness at both early time and late time, and effect a smooth transition between; hence, they are called *doubly asymptotic*. Doubly asymptotic TIRs are much more robust than singly asymptotic TIRs, which approach exactness in either the early-time (high-frequency, short-wavelength) or late-time (low-frequency, long-wavelength) limit, but not both (Mathews and Geers, 1987).

THREE-DIMENSIONAL TRANSIENT ELASTODYNAMICS

In the absence of body forces, the radiated/scattered displacement field in an infinite, isotropic, elastic medium may be expressed in terms of a scalar potential $\phi(\vec{x}, t)$ and a vector potential $\vec{\psi}(\vec{x}, t)$ as the Helmholtz decomposition (Eringen and Suhubi, 1975)

$$\vec{u}(\vec{x}, t) = \vec{\nabla}\phi(\vec{x}, t) + \vec{\nabla} \times \vec{\psi}(\vec{x}, t), \quad \vec{\nabla} \cdot \vec{\psi}(\vec{x}, t) = 0. \tag{1}$$

With this decomposition, the displacement equation of elastodynamics separates into the uncoupled wave equations

$$c_D^2 \nabla^2 \phi(\vec{x}, t) = \ddot{\phi}(\vec{x}, t), \quad c_S^2 \nabla^2 \vec{\psi}(\vec{x}, t) = \ddot{\vec{\psi}}(\vec{x}, t), \tag{2}$$

where an overdot denotes a time derivative, and c_D and c_S are the dilatational and shear phase velocities, respectively, given in terms of the Lamé parameters λ, μ and the mass density ρ by

$$c_D^2 = (\lambda + 2\mu)/\rho, \quad c_S^2 = \mu/\rho. \tag{3}$$

The associated stress tensor may be expressed

$$\overleftrightarrow{\sigma}(\vec{x}, t) = \lambda \nabla^2 \phi(\vec{x}, t) \overleftrightarrow{I} + 2\mu \nabla\nabla\phi(\vec{x}, t) + \mu\{\vec{\nabla}[\vec{\nabla} \times \vec{\psi}(\vec{x}, t)] + [\vec{\nabla} \times \vec{\psi}(\vec{x}, t)]\vec{\nabla}\}, \tag{4}$$

where $\overleftrightarrow{I}$ is the identity tensor.

An exact formula for both $\phi(\vec{X}, t)$ and $\vec{\psi}(\vec{X}, t)$, where $\vec{X}$ is any point on the operator boundary, is Kirchoff's retarded-potential formula (Eringen and Suhubi, 1975)

$$2\pi\varphi(\vec{X}, t) = \int_S \left\{R^{-1} \frac{\partial \varphi}{\partial n'}(\vec{X}', t_R) + R^{-2} \frac{\partial R}{\partial n'}\left[\varphi(\vec{X}', t_R) + \frac{R}{c}\dot{\varphi}(\vec{X}', t_R)\right]\right\} dS', \tag{5}$$

in which φ is either ϕ or $\vec{\psi}$ and c is either c_D or c_S, $R = |\vec{X} - \vec{X}'|$, n' is the boundary normal at $\vec{X}'$ (defined positive going into the elastic domain), and the retarded time $t_R = t - R/c$; $\partial(\,)/\partial n' \equiv \mathbf{e}'_n \cdot \vec{\nabla}(\,)$, where $\vec{e}'_n(\vec{X})$ is the unit normal on the boundary. An exact TIR that directly links the boundary displacement vector $\vec{u}(\vec{X}, t)$ and the boundary traction vector $\vec{t}(\vec{X}, t)$ is Love's integral identity, which may be written in Laplace-transform space for a smooth surface as (Cruse and Rizzo, 1968)

$$\frac{1}{2}\vec{u}(\vec{X}, s) + \int_S \vec{u}(\vec{X}', s)\overleftrightarrow{T}(\vec{X}, \vec{X}', s)\, dS' = \int_S \vec{t}(\vec{X}', s)\overleftrightarrow{U}(\vec{X}, \vec{X}', s)\, dS', \tag{6}$$

where $\overleftrightarrow{T}(\vec{X}, \vec{X}', s)$ and $\overleftrightarrow{U}(\vec{X}, \vec{X}', s)$ are Laplace-transformed dynamic Green's tensors. As mentioned above, utilization of (5) or (6) in engineering practice is currently impractical.

EARLY-TIME APPROXIMATIONS ETA$_1$ AND ETA$_2$

An analysis of (5) at early time (Felippa, 1980a) for a radiated or scattered field on a smooth boundary yields the *curved-wave approximation*

$$\dot{\varphi}(\vec{X}, t) + c\kappa(\vec{X})\varphi(\vec{X}, t) \approx -c\frac{\partial \varphi}{\partial n}(\vec{X}, t), \tag{7}$$

where $\kappa(\vec{X})$ is the mean curvature of the boundary (defined positive over a convex region) and n is the boundary normal at $\vec{X}$. Now (7) is also produced by ray theory (Nicolas-Vullierme, 1991) through the Laplace-transform representation (Keller, 1964)

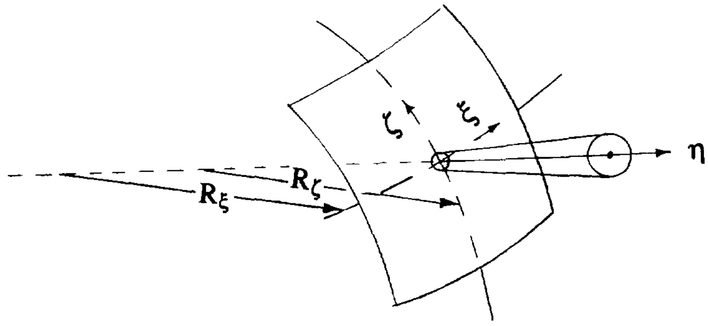

Fig. 1. Local curvilinear coordinate system and outward-travelling ray bundle.

$$\tilde{\varphi}(\xi, \zeta, \eta, s) \approx e^{-s\eta/c} \sqrt{\frac{R_\xi R_\zeta}{(\eta + R_\xi)(\eta + R_\zeta)}}\, \tilde{\varphi}(\xi, \zeta, 0, s), \tag{8}$$

where, as shown in Fig. 1, ξ, ζ, and η define a local curvilinear coordinate system at $\vec{\mathbf{X}}$, and $R_\xi(\vec{\mathbf{X}})$ and $R_\zeta(\vec{\mathbf{X}})$ are the principal radii of curvature (defined positive over a convex region). From (8), we find

$$\frac{\partial \varphi}{\partial n}(\vec{\mathbf{X}}, s) = \lim_{\eta \to 0} \frac{\partial \varphi}{\partial \eta}(\xi, \zeta, \eta, s) \approx -\left(\frac{s}{c} + \frac{R_\xi + R_\zeta}{2R_\xi R_\zeta}\right)\varphi(\vec{\mathbf{X}}, s), \tag{9}$$

which, upon inversion, agrees with (7).

Now a requirement for the applicability of elastodynamic ray theory is that the derivatives of ϕ and $\vec{\psi}$ in the normal direction greatly overshadow their counterparts in the tangential directions, e.g., $|\partial\phi/\partial\xi| \ll |\partial\phi/\partial\eta|$ and $|\partial\phi/\partial\zeta| \ll |\partial\phi/\partial\eta|$. Thus, for Fig. 1 we write $\xi = R_\xi\theta_\xi$ and $\zeta = R_\zeta\theta_\zeta$ to obtain the metric coefficients $h_\xi = R_\xi$, $h_\zeta = R_\zeta$ and $h_\eta = 1$ (Moon and Spencer, 1981), and thereby obtain the ray-theory boundary approximations

$$\vec{\nabla}\phi \approx \vec{\mathbf{e}}_n \frac{\partial \phi}{\partial n}, \quad \nabla^2 \phi \approx \frac{\partial^2 \phi}{\partial n^2} + 2\kappa \frac{\partial \phi}{\partial n},$$

$$\vec{\nabla} \times \vec{\psi} \approx -\vec{\mathbf{e}}_\xi \left(\frac{\partial \psi_\zeta}{\partial n} + R_\zeta^{-1}\psi_\zeta\right) + \vec{\mathbf{e}}_\zeta \left(\frac{\partial \psi_\xi}{\partial n} + R_\xi^{-1}\psi_\xi\right),$$

$$\nabla^2 \vec{\psi} \approx \vec{\mathbf{e}}_\xi \left(\frac{\partial^2 \psi_\xi}{\partial n^2} + 2\kappa \frac{\partial \psi_\xi}{\partial n} + \frac{R_\zeta - R_\xi}{R_\xi^2 R_\zeta}\psi_\xi\right) + \vec{\mathbf{e}}_\zeta \left(\frac{\partial^2 \psi_\zeta}{\partial n^2} + 2\kappa \frac{\partial \psi_\zeta}{\partial n} + \frac{R_\xi - R_\zeta}{R_\zeta^2 R_\xi}\psi_\zeta\right),$$

$$\text{Nonzero component of } \vec{\nabla}\vec{\nabla}\phi: \quad (\vec{\nabla}\vec{\nabla}\phi)_{nn} \approx \frac{\partial^2 \phi}{\partial n^2},$$

$$\text{Nonzero components of } \vec{\nabla}(\vec{\nabla} \times \vec{\psi}) + (\vec{\nabla} \times \vec{\psi})\vec{\nabla}:$$

$$[\vec{\nabla}(\vec{\nabla} \times \vec{\psi}) + (\vec{\nabla} \times \vec{\psi})\vec{\nabla}]_{\eta\xi} = [\vec{\nabla}(\vec{\nabla} \times \vec{\psi}) + (\vec{\nabla} \times \vec{\psi})\vec{\nabla}]_{\xi\eta} \approx -\frac{\partial^2 \psi_\zeta}{\partial n^2} - R_\zeta^{-1}\frac{\partial \psi_\zeta}{\partial n} + R_\zeta^{-2}\psi_\zeta,$$

$$[\vec{\nabla}(\vec{\nabla} \times \vec{\psi}) + (\vec{\nabla} \times \vec{\psi})\vec{\nabla}]_{\eta\zeta} = [\vec{\nabla}(\vec{\nabla} \times \vec{\psi}) + (\vec{\nabla} \times \vec{\psi})\nabla]_{\zeta\eta} \approx \frac{\partial^2 \psi_\xi}{\partial n^2} + R_\xi^{-1}\frac{\partial \psi_\xi}{\partial n} - R_\xi^{-2}\psi_\xi. \tag{10}$$

The ray-theory approximations enable us to derive the early-time approximation ETA_2 for elastodynamics. First, we obtain the displacement-potential relations by introducing the first and third of (10) into the first of (1), which yields

$$
\begin{aligned}
u_\eta &\approx \frac{\partial\phi}{\partial n},\\
u_\xi &\approx -\frac{\partial\psi_\zeta}{\partial n} - R_\xi^{-1}\psi_\zeta,\\
u_\zeta &\approx \frac{\partial\psi_\xi}{\partial n} + R_\zeta^{-1}\psi_\xi.
\end{aligned}
\tag{11}
$$

Next, we obtain the traction-potential relations by introducing the appropriate expressions from (10) into (4) and observe that the boundary tractions t_η, t_ξ and t_ζ are the negatives of $\sigma_{\eta\eta}$, $\sigma_{\eta\xi}$ and $\sigma_{\eta\zeta}$ there; this yields

$$
\begin{aligned}
t_\eta &\approx -\left(\rho c_D^2\frac{\partial^2\phi}{\partial n^2} + 2\kappa\lambda\frac{\partial\phi}{\partial n}\right),\\
t_\xi &\approx \rho c_S^2\left(\frac{\partial^2\psi_\zeta}{\partial n^2} + R_\xi^{-1}\frac{\partial\psi_\zeta}{\partial n} - R_\xi^{-2}\psi_\zeta\right),\\
t_\zeta &\approx -\rho c_S^2\left(\frac{\partial^2\psi_\xi}{\partial n^2} + R_\zeta^{-1}\frac{\partial\psi_\xi}{\partial n} - R_\zeta^{-2}\psi_\xi\right).
\end{aligned}
\tag{12}
$$

Then we obtain the approximate wave equations in the local coordinate system by introducing the second and fourth of (10) into (2), which yields

$$
\begin{aligned}
c_D^{-2}\ddot{\phi} &\approx \frac{\partial^2\phi}{\partial n^2} + 2\kappa\frac{\partial\phi}{\partial n},\\
c_S^{-2}\ddot{\psi}_\xi &\approx \frac{\partial^2\psi_\xi}{\partial n^2} + 2\kappa\frac{\partial\psi_\xi}{\partial n} + \frac{R_\xi - R_\zeta}{R_\xi^2 R_\zeta}\psi_\xi,\\
c_S^{-2}\ddot{\psi}_\zeta &\approx \frac{\partial^2\psi_\zeta}{\partial n^2} + 2\kappa\frac{\partial\psi_\zeta}{\partial n} + \frac{R_\zeta - R_\xi}{R_\zeta^2 R_\xi}\psi_\zeta.
\end{aligned}
\tag{13}
$$

The early-time relations (7), (11), (12) and (13) may be manipulated as described in the appendix to obtain ETA_2 in local coordinates:

$$
\dot{\vec{\mathbf{t}}}'(\vec{\mathbf{X}},t) + \kappa(\vec{\mathbf{X}})\vec{C}'\vec{\mathbf{t}}'(\vec{\mathbf{X}},t) = \rho\vec{C}'\ddot{\vec{\mathbf{u}}}'(\vec{\mathbf{X}},t) + \mu\kappa(\vec{\mathbf{X}})\vec{D}'\dot{\vec{\mathbf{u}}}'(\vec{\mathbf{X}},t),
\tag{14}
$$

where $\vec{\mathbf{t}}' = (t_\eta, t_\xi, t_\zeta)$, $\vec{\mathbf{u}}' = (u_\eta, u_\xi, u_\zeta)$, and $\vec{C}'$ and $\vec{D}'$ are diagonal matrices with nonzero elements c_D, c_S, c_S and 4, 2, 2, respectively. Thus, (14) consists of three uncoupled equations.

Finally, we employ the rotation tensor $\vec{Q}(\vec{\mathbf{X}})$ to transform from local to global Cartesian coordinates as

$$
\vec{\mathbf{u}}'(\vec{\mathbf{X}},t) = \vec{Q}(\vec{\mathbf{X}})\vec{\mathbf{u}}(\vec{\mathbf{X}},t), \quad \vec{\mathbf{t}}'(\vec{\mathbf{X}},t) = \vec{Q}(\vec{\mathbf{X}})\vec{\mathbf{t}}(\vec{\mathbf{X}},t).
\tag{15}
$$

Introducing these into (14), and then multiplying the result through by $\vec{Q}^{-1} = \vec{Q}^{\mathrm{T}}$, we obtain in global coordinates

$$
\mathrm{ETA}_2: \quad \dot{\vec{\mathbf{t}}}(\vec{\mathbf{X}},t) + \kappa(\vec{\mathbf{X}})\vec{C}(\vec{\mathbf{X}})\vec{\mathbf{t}}(\vec{\mathbf{X}},t) = \rho\vec{C}(\vec{\mathbf{X}})\ddot{\vec{\mathbf{u}}}(\vec{\mathbf{X}},t) + \mu\kappa(\vec{\mathbf{X}})\vec{D}(\vec{\mathbf{X}})\dot{\vec{\mathbf{u}}}(\vec{\mathbf{X}},t),
\tag{16}
$$

where $\vec{C}(\vec{\mathbf{X}}) = \vec{Q}^{\mathrm{T}}(\vec{\mathbf{X}})\vec{C}'\vec{Q}(\vec{\mathbf{X}})$ and $\vec{D}(\vec{\mathbf{X}}) = \vec{Q}^{\mathrm{T}}(\vec{\mathbf{X}})\vec{D}'\vec{Q}(\vec{\mathbf{X}})$. In contrast to (14), (16) consists of three coupled equations. Even so, ETA_2 is a spatially local approximation because all of the quantities are evaluated at a single point.

A lower-order ETA may be obtained from ETA_2 by noting that, at very early time, the first terms on either side of (16) overshadow their lower-derivative counterparts. Thus, neglecting the second terms on either side of (16) and integrating in time, we obtain in global coordinates (Underwood and Geers, 1981)

$$\mathrm{ETA}_1: \quad \vec{\mathbf{t}}(\vec{\mathbf{X}}, t) = \rho\vec{C}(\vec{\mathbf{X}})\dot{\mathbf{u}}(\vec{\mathbf{X}}, t). \tag{17}$$

Now, the preceding development makes no distinction between a body embedded in an infinite medium and one fully or partially embedded in a semi-infinite medium. No distinction is, in fact, necessary in the absence of early-time reflection of radiated/scattered waves from the free surface of the semi-infinite medium back to the body. For a body buried at depth comparable to a radius characterizing the body's size, this requirement is satisfied. However, in the case of a shallow-buried body, the medium between the structure and the free surface should be included in the domain contained within the integral-operator boundary, thereby creating a partially embedded radiator/scatterer. Then the only breakdown in the ETA model occurs in the small region near the line of contact between the body and the free surface, a region that is vanishingly small for vanishingly small time.

LATE-TIME APPROXIMATIONS LTA_1 AND LTA_2

The approximation LTA_2 is obtained by simply expanding the Green's tensors of (6) in Maclaurin series to get

$$\begin{aligned}\vec{T}(\vec{\mathbf{X}}, \vec{\mathbf{X}}', s) &= \vec{T}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}') + O(s^2), \\ \vec{U}(\vec{\mathbf{X}}, \vec{\mathbf{X}}', s) &= \vec{U}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}') + s\vec{U}^1(\vec{\mathbf{X}}, \vec{\mathbf{X}}') + O(s^2).\end{aligned} \tag{18}$$

Whether for a whole-space or a half-space, $\vec{T}^1(\vec{\mathbf{X}}, \vec{\mathbf{X}}') = 0$. The Cartesian-coordinate elements of $\vec{U}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$, $\vec{U}^1(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ and $\vec{T}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ for a whole-space are obtained by expanding the Green's tensors of Cruse and Rizzo, 1968, which yields

$$\begin{aligned}U_{ij}^0 &= \frac{K}{\mu R}[(3-4\nu)\delta_{ij} + R_{,i}R_{,j}] \\ U_{ij}^1 &= -\frac{4K}{3\mu}[(1-2\nu)/2c_D + 2(1-\nu)/c_S]\delta_{ij} \\ T_{ij}^0 &= \frac{2K}{R^2}\left\{[(1-2\nu)\delta_{ij} + 3R_{,i}R_{,j}]\frac{\mathrm{d}R}{\mathrm{d}n} + (1-2\nu)(R_{,i}n_j - R_{,j}n_i)\right\}\end{aligned} \tag{19}$$

where $K = 1/16\pi(1-\nu)$; we recall that $R = |\vec{\mathbf{X}} - \vec{\mathbf{X}}'|$ and n is positive going *into* the medium. The tensors $\vec{U}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ and $\vec{T}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ are, of course, Kelvin's elastostatics results (see Kane, 1994). The corresponding elements for a half-space are obtained by expanding the Green's tensors of Banerjee and Mamoon, 1990, which yields a long list of element formulae provided in Lewis, 1994. The tensors $\vec{U}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ and $\vec{T}^0(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ for the half-space are the elastostatics results in Mindlin, 1936.

Introducing (18) into (6), retaining terms of order s^0 and s^1, and inverse transforming, we obtain

$$\mathrm{LTA}_2: \quad \vec{A}(\vec{\mathbf{X}}, \vec{\mathbf{X}}')\dot{\mathbf{t}}(\vec{\mathbf{X}}', t) + \vec{B}(\vec{\mathbf{X}}, \vec{\mathbf{X}}')\vec{\mathbf{t}}(\vec{\mathbf{X}}', t) = \vec{\Gamma}(\vec{\mathbf{X}}, \vec{\mathbf{X}}')\vec{\mathbf{u}}(\vec{\mathbf{X}}', t), \tag{20}$$

in which the spatial operators $\mathbf{A}(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$, $\vec{B}(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ and $\vec{\Gamma}(\vec{\mathbf{X}}, \vec{\mathbf{X}}')$ are defined by

$$\vec{A}(\mathbf{X},\mathbf{X}')\dot{\mathbf{t}}(\mathbf{X}',t) \equiv \int_S \dot{\mathbf{t}}(\mathbf{X}',t)\vec{U}^1(\mathbf{X},\mathbf{X}')\,\mathrm{d}S', \quad \vec{B}(\mathbf{X},\mathbf{X}')\vec{\mathbf{t}}(\mathbf{X}',t) \equiv \int_S \vec{\mathbf{t}}(\mathbf{X}',t)\vec{U}^0(\mathbf{X},\mathbf{X}')\,\mathrm{d}S',$$

$$\vec{\Gamma}(\mathbf{X},\mathbf{X}')\vec{\mathbf{u}}(\mathbf{X}',t) \equiv \int_S \vec{\mathbf{u}}(\mathbf{X}',t)\left[\frac{1}{2}\delta(\vec{\mathbf{X}}-\vec{\mathbf{X}}')+\vec{T}^0(\mathbf{X},\mathbf{X}')\right]\mathrm{d}S' \qquad (21)$$

where $\delta(\vec{\mathbf{X}}-\vec{\mathbf{X}}')$ is the Dirac delta-function. We observe that these integral operators make LTA_2 spatially nonlocal.

A lower-order LTA may be obtained from LTA_2 by noting that, for slow motions at very late time, the second term on the left of (20) overshadows its higher-derivative companion. Thus, neglecting the first term on the left side of (20), we obtain (Underwood and Geers, 1981)

$$LTA_1: \quad \vec{B}(\vec{\mathbf{X}},\vec{\mathbf{X}}')\vec{\mathbf{t}}(\vec{\mathbf{X}}',t) = \vec{\Gamma}(\vec{\mathbf{X}},\vec{\mathbf{X}}')\vec{\mathbf{u}}(\vec{\mathbf{X}}',t). \qquad (22)$$

DOUBLY ASYMPTOTIC APPROXIMATIONS DAA_1, DAA_2 AND $DAA_{1\cdot2}$

DAAs may be systematically obtained by a procedure called *operator matching*, which is akin to solution matching in the method of matched asymptotic approximations (Van Dyke, 1964). A scalar form of the procedure was presented by Felippa, 1980b, extended to operator form in Nicolas-Vullierme, 1991 and refined in Geers and Zhang, 1994. In the following, the procedure for elastodynamic matching is described first for DAA_1, then for DAA_2, and finally for $DAA_{1\cdot2}$.

To begin the procedure for DAA_1, we Laplace-transform (17) and (22) to get

$$\vec{\mathbf{t}}(\vec{\mathbf{X}},s) = \rho\vec{C}(\vec{\mathbf{X}})s\vec{\mathbf{u}}(\vec{\mathbf{X}},s),$$

$$\vec{\mathbf{t}}(\vec{\mathbf{X}},s) = \vec{B}^{-1}(\vec{\mathbf{X}},\vec{\mathbf{X}}'')\vec{\Gamma}(\vec{\mathbf{X}}'',\vec{\mathbf{X}}')\vec{\mathbf{u}}(\vec{\mathbf{X}}',s), \qquad (23)$$

where $\vec{B}^{-1}$ is the operator-inverse of $\vec{\mathbf{B}}$. Next, we choose the DAA_1 trial equation

$$\vec{\mathbf{t}}(\vec{\mathbf{X}},s) = [s\vec{U}_1(\vec{\mathbf{X}},\vec{\mathbf{X}}')+\vec{U}_0(\vec{\mathbf{X}},\vec{\mathbf{X}}')]\vec{\mathbf{u}}(\vec{\mathbf{X}}',s), \qquad (24)$$

where the operators $\vec{U}_0$ and $\vec{U}_1$ are unknown, and then write it in the asymptotic forms

$$\vec{\mathbf{t}}(\vec{\mathbf{X}},s) = [\vec{U}_1(\vec{\mathbf{X}},\vec{\mathbf{X}}')+O(s^{-1})]s\vec{\mathbf{u}}(\vec{\mathbf{X}}',s), \quad s\to\infty,$$

$$\vec{\mathbf{t}}(\vec{\mathbf{X}},s) = [\vec{U}_0(\vec{\mathbf{X}},\vec{\mathbf{X}}')+O(s)]\vec{\mathbf{u}}(\vec{\mathbf{X}}',s), \quad s\to 0. \qquad (25)$$

Then we match the first of these to the first of (23) to obtain $\vec{U}_1(\vec{\mathbf{X}},\vec{\mathbf{X}}') = \rho\vec{C}(\vec{\mathbf{X}})\delta(\vec{\mathbf{X}}-\vec{\mathbf{X}}')$ and match the second to the second of (23) to obtain $\vec{U}_0(\vec{\mathbf{X}},\vec{\mathbf{X}}') = \vec{B}^{-1}(\vec{\mathbf{X}},\vec{\mathbf{X}}'')\vec{\Gamma}(\vec{\mathbf{X}}'',\vec{\mathbf{X}}')$. Finally, we introduce these results for $\vec{U}_0$ and $\vec{U}_1$ into (24) and inverse-transform the resulting equation to get (Underwood and Geers, 1981)

$$DAA_1: \quad \vec{\mathbf{t}}(\vec{\mathbf{X}},t) = \rho\vec{C}(\vec{\mathbf{X}})\dot{\vec{\mathbf{u}}}(\vec{\mathbf{X}},t)+\vec{B}^{-1}(\vec{\mathbf{X}},\vec{\mathbf{X}}'')\vec{\Gamma}(\vec{\mathbf{X}}'',\vec{\mathbf{X}}')\vec{\mathbf{u}}(\vec{\mathbf{X}}',t). \qquad (26)$$

We now employ the matching procedure to obtain DAA_2, omitting the spatial arguments in order to simplify notation. First, we Laplace-transform (16) and (20) to get

$$[1+s^{-1}\kappa\vec{C}]\vec{\mathbf{t}}(s) = [\rho\vec{C}+s^{-1}\mu\kappa\vec{D}]s\vec{\mathbf{u}}(s),$$

$$[\vec{B}+s\vec{A}]\vec{\mathbf{t}}(s) = \vec{\Gamma}\vec{\mathbf{u}}(s). \qquad (27)$$

Then we choose the DAA_2 trial equation [c.f. (24)]

$$[s\overleftrightarrow{I}+\overleftrightarrow{T}_0]\tilde{\mathbf{t}}(s) = [s^2\overleftrightarrow{U}_2+s\overleftrightarrow{U}_1+\overleftrightarrow{U}_0]\dot{\tilde{\mathbf{u}}}(s), \tag{28}$$

where the spatial operator $\overleftrightarrow{I}$ is defined by $\overleftrightarrow{I}(\vec{\mathbf{X}},\vec{\mathbf{X}}')\tilde{\mathbf{t}}(\vec{\mathbf{X}}',s) = \tilde{\mathbf{t}}(\vec{\mathbf{X}},s)$, and then write it in the asymptotic forms

$$[\overleftrightarrow{I}+s^{-1}\overleftrightarrow{T}_0]\dot{\tilde{\mathbf{t}}}(s) = [\overleftrightarrow{U}_2+s^{-1}\overleftrightarrow{U}_1+O(s^{-2})]s\dot{\tilde{\mathbf{u}}}(s), \quad s\to\infty,$$
$$[\overleftrightarrow{T}_0+s\overleftrightarrow{I}]\dot{\tilde{\mathbf{t}}}(s) = [\overleftrightarrow{U}_0+s\overleftrightarrow{U}_1+O(s^2)]\dot{\tilde{\mathbf{u}}}(s), \quad s\to 0, \tag{29}$$

To match in the limit $s\to\infty$, we multiply the first of (27) through by $1-s^{-1}\kappa\overleftrightarrow{C}$ and the first of (29) through by $\overleftrightarrow{I}-s^{-1}\overleftrightarrow{T}_0$ to obtain

$$[1+O(s^{-2})]\dot{\tilde{\mathbf{t}}}(s) = \{\rho\overleftrightarrow{C}+s^{-1}[\mu\kappa\overleftrightarrow{D}-\rho\kappa\overleftrightarrow{C}^2]+O(s^{-2})\}s\dot{\tilde{\mathbf{u}}}(s),$$
$$[1+O(s^{-2})]\dot{\tilde{\mathbf{t}}}(s) = \{\overleftrightarrow{U}_2+s^{-1}[\overleftrightarrow{U}_1-\overleftrightarrow{T}_0\overleftrightarrow{U}_2]+O(s^{-2})\}s\dot{\tilde{\mathbf{u}}}(s). \tag{30}$$

Matching these through order s^{-1}, we find

$$\overleftrightarrow{U}_2 = \rho\overleftrightarrow{C},$$
$$\overleftrightarrow{U}_1-\overleftrightarrow{T}_0\overleftrightarrow{U}_2 = \kappa[\mu\overleftrightarrow{D}-\rho\overleftrightarrow{C}^2]. \tag{31}$$

To match in the limit $s\to 0$, we multiply the second of (27) and the second of (29) through by $[\overleftrightarrow{I}-s\overleftrightarrow{B}^{-1}\overleftrightarrow{A}]\overleftrightarrow{B}^{-1}$ and $[\overleftrightarrow{I}-s\overleftrightarrow{T}_0^{-1}]\overleftrightarrow{T}_0^{-1}$, respectively, to get

$$[1+O(s^2)]\dot{\tilde{\mathbf{t}}}(s) = [1-s\overleftrightarrow{B}^{-1}\overleftrightarrow{A}]\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}\dot{\tilde{\mathbf{u}}}(s),$$
$$[1+O(s^2)]\dot{\tilde{\mathbf{t}}}(s) = \overleftrightarrow{T}_0^{-1}\{\overleftrightarrow{U}_0+s[\overleftrightarrow{U}_1-\overleftrightarrow{T}_0^{-1}\overleftrightarrow{U}_0]+O(s^2)\}\dot{\tilde{\mathbf{u}}}(s). \tag{32}$$

Matching these through order s, we obtain

$$\overleftrightarrow{T}_0^{-1}\overleftrightarrow{U}_0 = \overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma},$$
$$\overleftrightarrow{T}_0^{-1}[\overleftrightarrow{T}_0^{-1}\overleftrightarrow{U}_0-\overleftrightarrow{U}_1] = \overleftrightarrow{B}^{-1}\overleftrightarrow{A}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}. \tag{33}$$

Next, we solve (31) and (33), finding

$$\overleftrightarrow{T}_0 = \overleftrightarrow{\Omega}, \qquad \overleftrightarrow{U}_1 = \overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}-\overleftrightarrow{\Omega}\overleftrightarrow{B}^{-1}\overleftrightarrow{A}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma},$$
$$\overleftrightarrow{U}_0 = \overleftrightarrow{\Omega}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}, \quad \overleftrightarrow{U}_2 = \rho\overleftrightarrow{C}, \tag{34}$$

where $\overleftrightarrow{\Omega} = [\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}+\kappa(\rho\overleftrightarrow{C}^2-\mu\overleftrightarrow{D})][\rho\overleftrightarrow{C}+\overleftrightarrow{B}^{-1}\overleftrightarrow{A}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}]^{-1}$. Finally, we introduce these results into (28) and inverse-transform to get

$$\mathrm{DAA}_2: \quad \dot{\mathbf{t}}+\overleftrightarrow{\Omega}\mathbf{t} = \rho\overleftrightarrow{C}\ddot{\mathbf{u}}+(\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}-\overleftrightarrow{\Omega}\overleftrightarrow{B}^{-1}\overleftrightarrow{A}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma})\dot{\mathbf{u}}+\overleftrightarrow{\Omega}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}\mathbf{u}. \tag{35}$$

A substantially simpler approximation, $\mathrm{DAA}_{1\frac{1}{2}}$, is obtained by taking $\overleftrightarrow{A} = 0$, which amounts to matching the trial equation (28) to LTA_1 instead of LTA_2 in the limit $s\to 0$. The second of (33) still applies, however, yielding $\overleftrightarrow{T}_0^{-1}\overleftrightarrow{U}_0-\overleftrightarrow{U}_1 = 0$. Thus, (35) reduces to

$$\mathrm{DAA}_{1\frac{1}{2}}: \quad \dot{\mathbf{t}}+\overleftrightarrow{\Omega}\mathbf{t} = \rho\overleftrightarrow{C}\ddot{\mathbf{u}}+\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}\dot{\mathbf{u}}+\overleftrightarrow{\Omega}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}\mathbf{u}, \tag{36}$$

where $\overleftrightarrow{\Omega} = \rho^{-1}\overleftrightarrow{B}^{-1}\overleftrightarrow{\Gamma}\overleftrightarrow{C}^{-1}+\kappa(\overleftrightarrow{C}-c_S^2\overleftrightarrow{D}\overleftrightarrow{C}^{-1})$. Finally, we observe that, because their temporal operators are differential, DAAs are temporally local; however, because of the integral operators associated with late-time approximation, they are spatially nonlocal.

BOUNDARY ELEMENT DISCRETIZATION

Matrix forms of the early-time, late-time and doubly asymptotic approximations are readily obtained by boundary-element discretization (see Kane, 1994). With the boundary displacement and traction fields approximated as

$$\vec{u}(\vec{X}, t) = \mathbf{v}^{\mathrm{T}}(\vec{X})\vec{\mathbf{u}}(t), \quad \vec{t}(\vec{X}, t) = \mathbf{v}^{\mathrm{T}}(\vec{X})\vec{\mathbf{t}}(t), \tag{37}$$

where $\mathbf{v}(\vec{X})$ is an $N \times 3$ matrix of interpolation functions, and $\vec{\mathbf{u}}(t)$ and $\vec{\mathbf{t}}(t)$ are column vectors of nodal displacement and traction responses, application of the method of weighted residuals to (16) and (20) yields

$$\begin{aligned} \mathrm{ETA}_2: &\quad \mathbf{J}\dot{\vec{\mathbf{t}}} + \mathbf{K}\vec{\mathbf{t}} = \rho\mathbf{C}\ddot{\mathbf{u}} + \mathbf{M}\dot{\mathbf{u}}, \\ \mathrm{LTA}_2: &\quad \mathbf{A}\dot{\vec{\mathbf{t}}} + \mathbf{B}\vec{\mathbf{t}} = \mathbf{G}\vec{\mathbf{u}}, \end{aligned} \tag{38}$$

in which

$$\begin{aligned} \mathbf{J} &= \int_S \mathbf{w}(\vec{X})\mathbf{v}^{\mathrm{T}}(\vec{X})\,\mathrm{d}S, \quad \mathbf{C} = \int_S \mathbf{w}(\vec{X})C(\vec{X})\mathbf{v}^{\mathrm{T}}(\vec{X})\,\mathrm{d}S, \\ \mathbf{K} &= \int_S \mathbf{w}(\vec{X})\kappa(\vec{X})\vec{C}(\vec{X})\mathbf{v}^{\mathrm{T}}(\vec{X})\,\mathrm{d}S, \quad \mathbf{M} = \mu\int_S \mathbf{w}(\vec{X})\kappa(\vec{X})\vec{D}(\vec{X})\mathbf{v}^{\mathrm{T}}(\vec{X})\,\mathrm{d}S, \\ \mathbf{A} &= \int_S \mathbf{w}(\vec{X})\int_S \mathbf{v}^{\mathrm{T}}(\vec{X}')\vec{U}^1(\vec{X}, \vec{X}')\,\mathrm{d}S'\,\mathrm{d}S, \\ \mathbf{G} &= \int_S \mathbf{w}(\vec{X})\int_S \mathbf{v}^{\mathrm{T}}(\vec{X}')\left[\frac{1}{2}\delta(\vec{X}-\vec{X}') + \vec{T}^0(\vec{X}, \vec{X}')\right]\mathrm{d}S'\,\mathrm{d}S, \\ \mathbf{B} &= \int_S \mathbf{w}(\vec{X})\int_S \mathbf{v}^{\mathrm{T}}(\vec{X}')\vec{U}^0(\vec{X}, \vec{X}')\,\mathrm{d}S'\,\mathrm{d}S, \end{aligned} \tag{39}$$

where $\mathbf{w}(\vec{X})$ is an $N \times 3$ matrix of weighting functions. ETA_1 and LTA_1 are, of course, given by (38) with $\mathbf{K} = \mathbf{M} = 0$ and $\mathbf{A} = 0$.

Now a matrix DAA_1 may be obtained by multiplication of (26) through by $\vec{B}$, followed by direct boundary element discretization. A similar process cannot be used to obtain a matrix DAA_2 or $\mathrm{DAA}_{1\ 2}$, however, because of the undefined inverse operators that appear in those approximations. Thus, the appropriate procedure by which to obtain matrix DAAs is that of *matrix matching*, which is the discrete analog of operator matching, as used above. Starting with the Laplace transforms of (38), we can apply matrix matching to obtain [c.f. (35)]

$$\mathrm{DAA}_2: \quad \dot{\mathbf{t}} + \Omega\mathbf{t} = \rho\mathbf{J}^{-1}\mathbf{C}\ddot{\mathbf{u}} + (\mathbf{B}^{-1}\mathbf{G} - \Omega\mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{-1}\mathbf{G})\dot{\mathbf{u}} + \Omega\mathbf{B}^{-1}\mathbf{G}\vec{\mathbf{u}}, \tag{40}$$

where $\Omega = [\mathbf{B}^{-1}\mathbf{G} + \mathbf{J}^{-1}(\rho\mathbf{K}\mathbf{J}^{-1}\mathbf{C} - \mathbf{M})][\rho\mathbf{J}^{-1}\mathbf{C} + \mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{-1}\mathbf{G}]^{-1}$. The simpler approximation $\mathrm{DAA}_{1\ 2}$ is generated by taking $\mathbf{A} = 0$. The even simpler DAA_1 is produced by taking $\Omega = 0$.

CANONICAL PROBLEM

To provide some initial insight into the nature and accuracy of doubly asymptotic approximations for transient elastodynamics, we consider the simple problem of a spherical cavity with radius R in an infinite elastic domain loaded by a uniform, radial step-traction (pressure loading). For this problem, $\phi(\vec{x}, t) = \phi(r, t)$ and $\vec{\psi}(\vec{x}, t) = 0$, so the first of (1), the first of (2), and (4) become

$$u = \partial\phi/\partial r, \quad c_D^2\left(\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}\right) = \frac{\partial^2\phi}{\partial t^2}, \quad \sigma = \rho c_D^2\frac{\partial^2\phi}{\partial r^2}+\frac{2\lambda}{r}\frac{\partial\phi}{\partial r}. \tag{41}$$

Laplace transformation of the second of these yields an equation that possesses the following radiation-type solution:

$$\tilde{\phi}(r,s) = f(s)k_0(z), \tag{42}$$

where $k_0(z) = z^{-1}e^{-z}$ is the modified spherical Bessel function of order zero and $z = rs/c_D$ (Abramowitz and Stegun, 1964).

We can easily obtain the exact temporal impedance relation for this problem as follows. First, we eliminate the radial derivatives in the last of (41) by utilizing the first of and second of (41), which yields

$$\sigma(r,t) = \rho\frac{\partial^2\phi}{\partial t^2}(r,t)-\frac{4\mu}{r}u(r,t). \tag{43}$$

Then we take the Laplace transform of this equation, introduce (42), and let $z = Z = Rs/c_D$ to get

$$\tilde{\sigma}(R,s) = \left[\rho c_D s\frac{k_0(Z)}{k_0'(Z)}-\frac{4\mu}{R}\right]\tilde{u}(R,s), \tag{44}$$

where k_0' is the derivative of k_0 with respect to argument. Now it is readily found that $k_0(Z)/k_0'(Z) = -1/(1+Z^{-1})$; also, the radial displacement and traction at the surface of the cavity are $U(t) = u(R,t)$ and $P(t) = -\sigma(R,t)$, respectively. Thus, introducing the Laplace transforms of these relations into (44), multiplying through by $-s(1+Z^{-1})$, and inverse transforming, we obtain at the cavity surface the exact TIR

$$\dot{P}+\frac{c_D}{R}P = \rho c_D\ddot{U}+\frac{4\mu}{R}\dot{U}+\frac{4\mu c_D}{R^2}U. \tag{45}$$

At very early time, when the highest derivatives on either side dominate, this equation yields the plane-wave relation $P = \rho c_D\dot{U}$; at late time, when the lowest derivatives dominate, it yields the quasi-static relation $P = (4\mu/R)U$.

We can also obtain doubly asymptotic approximations. First, ETA$_2$ is found as follows. We introduce $P(t) = -\sigma(R,t)$ and $k_0(Z)/k_0'(Z) = -1/(1+Z^{-1})$ into (44), and multiply the result through by $-(1+Z^{-1})$ to get

$$(1+s^{-1}c_D/R)\tilde{P}(s) = \left[\rho c_D+\frac{4\mu}{R}(s^{-1}+s^{-2}c_D/R)\right]s\tilde{U}(R,s). \tag{46}$$

Then we retain only the first two terms in the bracket on the right, multiply the result through by s, and inverse-transform to obtain ETA$_2$ [cf. (16)]

$$\dot{P}+\frac{c_D}{R}P = \rho c_D\ddot{U}+\frac{4\mu}{R}\dot{U}. \tag{47}$$

We see that this produces ETA$_1$ at very early time.

Next, we find LTA$_2$ by expanding $k_0(Z)/k_0'(Z) = -Z/(1+Z)$ for small Z as $k_0(Z)/k_0'(Z) = -Z(1-Z+Z^2-\cdots)$, and introducing this, $\tilde{U}(s) = \tilde{u}(R,s)$ and $\tilde{P}(s) = -\tilde{\sigma}(R,s)$ into (44) to get

$$\tilde{P}(s) = \left[\frac{4\mu}{R} + O(s^2)\right]\tilde{U}(s). \tag{48}$$

Then we drop the higher-order terms on the right and inverse-transform to obtain LTA_2

$$P = \frac{4\mu}{R}U. \tag{49}$$

Comparing this with (20), we see that $\vec{A} = 0$ for this problem.

Finally, we use scalar matching, the scalar analog of the operator matching, to determine DAA_2. Following the procedure used to obtain (35), we find

$$\dot{P} + \Omega P = \rho c_D \ddot{U} + \frac{4\mu}{R}\dot{U} + \Omega\frac{4\mu}{R}U, \tag{50}$$

where $\Omega = c_D/R$. As this is equivalent to (45), we conclude that DAA_2 is exact for this problem. Also, because LTA_2 is the same as LTA_1 ($\vec{A} = 0$), $DAA_{1,2}$ is also exact for this problem.

For $P(t) = P_0 H(t)$, where $H(t)$ is the Heaviside step-function, (45) and (50) yield the Exact/DAA_2/$DAA_{1,2}$ solution

$$U(t) = \frac{P_o R}{4\mu}\left[(1 - e^{-\zeta_0\hat{\omega}_0\hat{t}}\cos\hat{\Omega}_0\hat{t}) + \frac{1}{2}\hat{\omega}_0^2\hat{\Omega}_0^{-1}e^{-\zeta_0\hat{\omega}_0\hat{t}}\sin\hat{\Omega}_0\hat{t}\right], \tag{51}$$

where $\hat{\omega}_0 = 2c_S/c_D$, $\zeta_0 = c_S/c_D$, $\hat{\Omega}_0 = 2(1-\zeta_0^2)^{1/2}c_S/c_D$ and $\hat{t} = c_D t/R$. A more complicated form of this solution appears in Timoshenko and Goodier, 1970. In contrast, (50) with $\Omega = 0$ gives the DAA_1 solution

$$U(t) = \frac{P_o R}{4\mu}(1 - e^{-\hat{\omega}_0^2\hat{t}}), \tag{52}$$

(47) gives the ETA_2 solution

$$U(t) = \frac{P_0 R}{4\mu}[(1 - \hat{\omega}_0^{-2})(1 - e^{-\hat{\omega}_0^2\hat{t}}) + \hat{t}], \tag{53}$$

and (49) gives the LTA_2 solution

$$U(t) = \frac{P_o R}{4\mu}H(\hat{t}). \tag{54}$$

Numerical results produced by (51)–(54) for $\nu = 1/3$, which produces

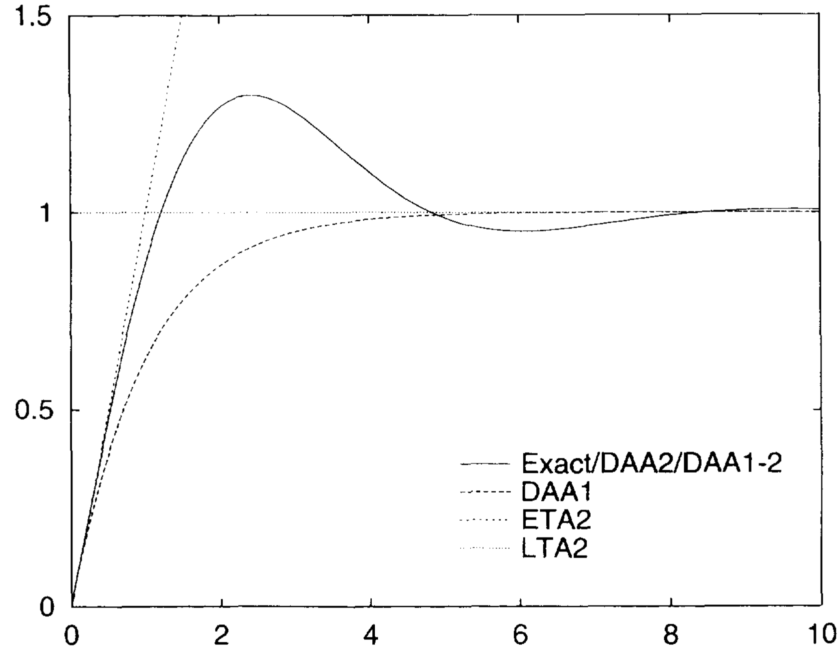

Fig. 2. Radial displacement of a step-pressurized spherical cavity in an infinite, isotropic, elastic medium: exact/DAA_2/DAA_{1-2}, DAA_1, ETA_2 and LTA_2 solutions.

$c_S/c_D = [(1-2\nu)/2(1-\nu)]^{1/2} = 1/2$, are shown in Fig. 2. ETA_2, which yields the same result that ETA_1 does for $c_S/c_D = 1/2$, succeeds at early time but fails at late time. LTA_2, which is the same as LTA_1 for this problem, fails at early time but succeeds at late time. DAA_1 succeeds at both early and late times, but misses the overshoot at intermediate time. More comprehensive DAA evaluations appear in Geers *et al.*, 1997a and 1997b; they disclose excellent DAA_2 and DAA_{1-2} performance.

CONCLUSION

First- and second-order, singly and doubly asymptotic approximations have been formulated for the transient response analysis of a body embedded in an infinite or semi-infinite, uniform, isotropic, elastic medium. Only DAA_{1-2} and DAA_2 appear sufficiently robust to handle problems involving broad-band excitations. Higher-order DAAs are currently being formulated for computational acoustics. A complicating factor in these extensions is the appearance of tangential field derivatives in ETA_3 (Geers, 1991).

Acknowledgements—This study was sponsored under Contract DNA001-92-C-0038 with the Defense Nuclear Agency, Alexandria, VA. The authors thank Mr Michael Giltrud and Dr Kent Goering for their encouragement and support.

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APPENDIX: DERIVATION OF ETA$_2$

We work in this appendix with the Laplace transforms of (7) and (11)–(13), omitting repetitive statements regarding Laplace transformation. We also simplify notation by using a prime to denote a partial derivative with respect to the normal n.

Observing in (7) and (11)–(13) that the equations involving the scalar potential ϕ are uncoupled from those involving the vector potential $\vec{\psi}$, we first derive ETA$_2$ for dilatational waves. Employing the first of (13) to eliminate $\tilde{\phi}''$ in the first of (12), and subsequently using (7) with $\varphi = \phi$ to eliminate $\tilde{\phi}$ itself, we get

$$\tilde{t}_n = [\rho s^2(\kappa + s/c_D)^{-1} + 4\mu\kappa]\tilde{\phi}'. \tag{A1}$$

Then we utilize the first of (11) to eliminate $\tilde{\phi}'$ and multiply the resulting equation through by $s+\kappa c_D$ we obtain

$$(s+\kappa c_D)\tilde{t}_n = (\rho c_D s^2 + 4\mu\kappa s + 4\mu\kappa^2 c_D)\tilde{u}_n. \tag{A2}$$

Finally, because ETA$_2$ requires only two terms on the left and two terms on the right, we drop the third term on the right side of (A2) and inverse-transform to arrive at the first of (14).

The derivation of ETA$_2$ for shear waves is more complicated. First, we use (7) with $\varphi = \vec{\psi}$ to eliminate $\tilde{\psi}'_\xi$ and $\tilde{\psi}'_\zeta$ in the last two of (12), which yields

$$\begin{aligned} \tilde{t}_\xi &= \rho c_S^2[\tilde{\psi}''_\zeta - R_\xi^{-1}(\kappa + R_\zeta^{-1} + s/c_S)\tilde{\psi}_\zeta], \\ \tilde{t}_\zeta &= -\rho c_S^2[\tilde{\psi}''_\xi - R_\zeta^{-1}(\kappa + R_\xi^{-1} + s/c_S)\tilde{\psi}_\xi]. \end{aligned} \tag{A3}$$

Second, we similarly eliminate $\tilde{\psi}'_\xi$ and $\tilde{\psi}'_\zeta$ in the last two of (13) to get

$$\begin{aligned} \tilde{\psi}''_\xi &= [(s/c_S)^2 + 2\kappa(\kappa + s/c_S) - 2\tau_\xi]\tilde{\psi}_\xi, \\ \tilde{\psi}''_\zeta &= [(s/c_S)^2 + 2\kappa(\kappa + s/c_S) - 2\tau_\zeta]\tilde{\psi}_\zeta, \end{aligned} \tag{A4}$$

where $\tau_\xi = R_\xi^{-1}(R_\zeta^{-1} - R_\xi^{-1})/2$ and $\tau_\zeta = R_\zeta^{-1}(R_\xi^{-1} - R_\zeta^{-1})/2$. Third, we eliminate the second derivatives in (A3) by introducing (A4) into them, which yields

$$\begin{aligned} \tilde{t}_\xi &= \rho c_S^2[(s/c_S)^2 + (2\kappa - R_\xi^{-1})(s/c_S) + 2(\kappa^2 - \tau_\zeta) - R_\xi^{-1}(\kappa + R_\zeta^{-1})]\tilde{\psi}_\zeta, \\ \tilde{t}_\zeta &= -\rho c_S^2[(s/c_S)^2 + (2\kappa - R_\zeta^{-1})(s/c_S) + 2(\kappa^2 - \tau_\xi) - R_\zeta^{-1}(\kappa + R_\xi^{-1})]\tilde{\psi}_\xi. \end{aligned} \tag{A5}$$

Fourth, we use the Laplace transform of (7) with $\varphi = \vec{\psi}$ to eliminate $\tilde{\psi}'_\xi$ and $\tilde{\psi}'_\zeta$ in the last two of (11), thereby finding

$$\begin{aligned} \tilde{u}_\xi &= (\kappa - R_\xi^{-1} + s/c_S)\tilde{\psi}_\zeta, \\ \tilde{u}_\zeta &= -(\kappa - R_\zeta^{-1} + s/c_S)\tilde{\psi}_\xi. \end{aligned} \tag{A6}$$

Fifth, we eliminate $\tilde{\psi}_\xi$ and $\tilde{\psi}_\zeta$ from (A5) by employing (A6), which gives

$$(\kappa - R_\zeta^{-1} + s/c_S)\tilde{t}_\xi = \rho c_S^2[(s/c_S)^2 + (2\kappa - R_\zeta^{-1})(s/c_S) + 2(\kappa^2 - \tau_\xi) - R_\zeta^{-1}(\kappa + R_\xi^{-1})]\tilde{u}_\xi,$$

$$(\kappa - R_\xi^{-1} + s/c_S)\tilde{t}_\zeta = \rho c_S^2[(s/c_S)^2 + (2\kappa - R_\xi^{-1})(s/c_S) + 2(\kappa^2 - \tau_\zeta) - R_\xi^{-1}(\kappa + R_\zeta^{-1})]\tilde{u}_\zeta. \quad \text{(A7)}$$

The coefficients in (A7) are readily simplified. Recalling that $\kappa = (R_\xi^{-1} + R_\zeta^{-1})$, $\tau_\xi = R_\xi^{-1}(R_\zeta^{-1} - R_\xi^{-1})/2$ and $\tau_\zeta = R_\zeta^{-1}(R_\xi^{-1} - R_\zeta^{-1})/2$, and defining $\beta = (R_\xi^{-1} - R_\zeta^{-1})/2$, we find

$$\kappa - R_\zeta^{-1} = \beta, \quad 2\kappa - R_\zeta^{-1} = R_\xi^{-1}, \quad 2(\kappa^2 - \tau_\xi) - R_\zeta^{-1}(\kappa + R_\xi^{-1}) = \beta R_\zeta^{-1},$$

$$\kappa - R_\xi^{-1} = -\beta, \quad 2\kappa - R_\xi^{-1} = R_\zeta^{-1}, \quad 2(\kappa^2 - \tau_\zeta) - R_\xi^{-1}(\kappa + R_\zeta^{-1}) = -\beta R_\xi^{-1}. \quad \text{(A8)}$$

Thus, (A7) become

$$(s + \beta c_S)\tilde{t}_\xi = \rho c_S(s^2 + R_\xi^{-1}c_S s + \beta R_\zeta^{-1})\tilde{u}_\xi,$$

$$(s - \beta c_S)\tilde{t}_\zeta = \rho c_S(s^2 + R_\zeta^{-1}c_S s - \beta R_\xi^{-1})\tilde{u}_\zeta. \quad \text{(A9)}$$

Sixth, we multiply the first of these through by $(s+\kappa c_S)/(s+\beta c_S)$ and the second through by $(s+\kappa c_S)/(s-\beta c_S)$ to get

$$(s + \kappa c_S)\tilde{t}_\xi = \rho c_S(s^2 - 2\kappa c_S s + O(s^0))]\tilde{u}_\xi,$$

$$(s + \kappa c_S)\tilde{t}_\zeta = \rho c_S(s^2 + 2\kappa c_S s + O(s^0))]\tilde{u}_\zeta. \quad \text{(A10)}$$

Finally, we drop the $O(s^0)$ terms in these equations and inverse-transform to arrive at the last two of (14).